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2006 J. Phys. A: Math. Gen. 39 559

(http://iopscience.iop.org/0305-4470/39/3/008)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 559–564 doi:10.1088/0305-4470/39/3/008

Quantum back-reaction and the particle law of motion

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Received 26 October 2005, in final form 27 October 2005
Published 21 December 2005
Online at stacks.iop.org/JPhysA/39/559

Abstract
In a variational formulation of the interaction of a point particle with the spin 0 Schrödinger field in which there is no back-reaction of the particle on the wave, it is shown that the condition that (a certain subset of the) components of the current and energy–momentum complexes of the composite take their quantum values fixes the particle law of motion as that given by de Broglie and Bohm. No appeal to statistics is made.

PACS number: 03.65.Ta

1. Introduction

Within the community of researchers in de Broglie–Bohm and similar theories, in which a physical corpuscle pursuing a deterministic spacetime path is added to the wavefunction, remarkably little attention has been paid to justifying the particle law of motion. This may seem strange in a theory whose principal conceptual boast is its unambiguous ontology and whose most basic law presumably represents an actual process. A plethora of theoretically acceptable laws would surely raise questions about ontological consistency.

The original argument sought a flow that generates the quantal statistical distribution $|\psi(t)|^2$ given the initial distribution $|\psi_0|^2$ [1]. Writing the wavefunction in polar form, $\psi = \sqrt{\rho}\exp(\mathrm{i}S/\hbar)$, in the one-body case this led to the proposal

$$m\frac{\mathrm{d}q_i(t)}{\mathrm{d}t} = \left.\frac{\partial S(x,t)}{\partial x_i}\right|_{x_i=q_i(t)}, \qquad i = 1, 2, 3, \tag{1.1}$$

where m is the mass of the particle. We call this the de Broglie–Bohm law of motion.

Unfortunately, the statistical requirement is too weak to make (1.1) the unique option in the infinite class of conceivable laws that are compatible with the quantal distribution. This is not surprising; it is akin to attempting to derive an individual classical molecule's law of motion from statistical mechanics. Attempts have been made to restrict the choices. An appeal to 'simplicity' is countered by the existence of other deterministic theories that generate the required flow and are otherwise acceptable. Indeed, (1.1) is wrong in the case of the electron where compatibility with relativity requires a modification of the right-hand side [2] (so that (1.1) applies only to spin 0). The latter argument proves uniqueness for a flow that generates the statistical distribution directly and does not exclude flows with a less direct connection with probability [3].

The question arises whether we can dispense with the statistical argument altogether and find a physical basis for adopting the de Broglie–Bohm law. Here, we attempt to develop an alternative method suggested by consideration of the dynamical implications and consistency of one of the most distinctive aspects of the de Broglie–Bohm theory: that, in acting on the particle, the guiding wave suffers no back-reaction. This property is crucial if one aims to avoid disturbing the Schrödinger evolution and hence maintain fully the usual predictions of quantum mechanics (which are of course entirely independent of the corpuscle variables). This is not a logical problem as there is no principle of physics that requires such reaction in all cases but it is nevertheless non-trivial to derive an appropriate dynamical framework for the particle–wave interaction displaying this feature. The nature of the problem may be appreciated within the hydrodynamical model, in which the wavefunction is represented by Eulerian fluid functions and the corpuscle is regarded as a foreign body immersed in the fluid [4]. In the analogous classical problem, where the mutual actions of the contaminant and the fluid obey Newtonian principles, it is expected that the body will be passive when, among other things, its mass is small relative to that of a fluid element [5]. In contrast, the quantum case presents certain novelties which necessitate a different approach: the corpuscle moves along the track of a fluid particle and exerts no influence on the fluid, even at the location of the fluid element it displaces, yet it has a mass far greater than the mass of the element.

An approach to the back-reaction problem has been given previously in terms of a canonical formulation of the wave–particle interaction that involves the introduction of auxiliary fields [6, 7]. This established a Hamiltonian framework for the interaction that is compatible with the canonical theory of the wave equation and clarified the connection between the de Broglie–Bohm and Hamilton–Jacobi theories. In that work, the quantum effects on the particle were attributed to the quantum potential (and the initial conditions required to generate a quasi-potential flow). Here, we shall solve a somewhat different and more general problem. We first allow within a canonical theory of interaction a much broader dependence of the potential on ψ than is exhibited by the quantum potential. It is then necessary to consider consistency conditions on the wave–particle composite that constrain its elements and their interactions. For example, when the composite interacts with another physical system, that system should not 'see' more (empirically) than a 'quantum system', i.e., the composite should behave like the bare Schrödinger field (regarded as a classical field). This can be achieved if key functions characterizing the composite coincide with the corresponding quantum functions. The functions to which we shall apply this idea are those for which the composite is the 'source' of another system—of an electromagnetic field through the current it generates, for instance, or of a gravitational field through its energy–momentum complex (regarded as the low-energy limit of some relativistic system). On the basis of quite mild constraints, we find as a concomitant of the conditions for the composite to behave as a 'quantum system' that the particle variables must obey the de Broglie–Bohm law (1.1). Specifically, we require that the zeroth component of the current and the energy and momentum densities of the composite coincide with their Schrödinger counterparts. This appears to be the first time that a justification for the de Broglie–Bohm law has been given that is independent of statistical arguments.

2. Quantum wave–particle interaction without particle back-reaction

It is convenient to represent the wavefunction $\psi(x,t)$ by the two real fields $\rho(x,t)$ and $S(x,t)$ defined in section 1. In the canonical formulation of the theory considered here, the latter are regarded as coordinates rather than conjugate variables as is usually assumed. We develop the theory in this way so that including the fields in the particle component of the total Lagrangian does not modify Schrödinger's equation through the appearance of a (singular) particle source term. This approach necessitates the introduction of additional field variables $g_1(x,t)$ and $g_2(x,t)$ whose coupled evolution equations do include source terms.

Denoting the particle variables by $q(t)$, the total Lagrangian is

$$L(q,\dot{q},\rho,S,t) = \frac{1}{2}m\dot{q}_i^2 - V(q,t) - V_Q(\psi(q,t)) + \int \left\{ g_1\left(\dot{\rho} + \frac{1}{m}\partial_i(\rho\partial_i S)\right) + g_2\left(\dot{S} + \frac{1}{2m}(\partial_i S)^2 + Q + V(x,t)\right)\right\} \mathrm{d}^3x, \tag{2.1}$$

where $\dot{\rho} = \partial\rho(x,t)/\partial t$ etc, $\partial_i \equiv \partial/\partial x_i$, $Q = (-\hbar^2/2m\sqrt{\rho})\partial_i^2\sqrt{\rho}$ is the quantum potential, V is the external potential and we represent the quantum effects on the particle through the scalar potential V_Q. The latter is assumed to depend (locally) on ρ and S but not on g_1 and g_2 so that Schrödinger's equation is unmodified. The theory is also assumed to be globally gauge invariant, which is ensured if V_Q is a gauge scalar. The most general dependence of V_Q is then on ρ and its derivatives and on the derivatives of S. Since our principal aim is to illustrate the method, we shall also assume that V_Q is independent of the time derivatives of the fields in order to avoid complications in defining the canonical energy and momentum of the composite needed later (the method can in principle be extended to the more general case).

Variation of (2.1) with respect to g_1 and g_2 yields, respectively, the equations

$$\frac{\partial\rho}{\partial t} + \frac{1}{m}\partial_i(\rho\partial_i S) = 0 \tag{2.2}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\partial_i S)^2 + Q + V = 0. \tag{2.3}$$

These equations are equivalent to the Schrödinger equation if the fields obey conditions corresponding to those imposed on ψ (single-valuedness, boundedness, etc). Next, varying with respect to ρ and S gives, in turn,

$$\frac{\partial g_1}{\partial t} + \frac{1}{m}\partial_i g_1 \partial_i S = \frac{\delta}{\delta\rho(x)}\int g_2(x',t)Q(\rho(x',t))\,\mathrm{d}x' - \frac{\delta V_Q}{\delta\rho(x)} \tag{2.4}$$

$$\frac{\partial g_2}{\partial t} - \frac{1}{m}\partial_i(\rho\partial_i g_1 - g_2\partial_i S) = -\frac{\delta V_Q}{\delta S(x)}. \tag{2.5}$$

As expected, the field equations for ρ and S are unmodified by the particle variables whereas the equations for the auxiliary fields g_1 and g_2 are (the right-hand sides of (2.4) and (2.5) involve the function $\delta(x - q(t))$ and its derivatives). In fact, the two auxiliary equations are equivalent to the Schrödinger equation with a source term, for a 'wavefunction' defined in terms of ρ, S, g_1 and g_2, and the complex conjugate equation [7]. Finally, varying the variables q, we obtain

$$m\ddot{q}_i = -\frac{\partial}{\partial q_i}(V + V_Q)\bigg|_{q=q(t)}. \tag{2.6}$$

The components of the system current (the Noether current associated with the gauge symmetry) are

$$J_0(x,q,t) = \delta(x - q(t)) - g_2 \tag{2.7}$$

and

$$J_i(x,q,t) = \dot{q}_i\delta(x-q(t)) + \frac{1}{m}\rho\partial_i g_1 - \frac{1}{m}g_2\partial_i S - \Delta_i, \tag{2.8}$$

where

$$\Delta_i = -\frac{\partial V_Q(x)\delta(x-q)}{\partial(\partial S/\partial x_i)} + \frac{\partial}{\partial x_j}\frac{\partial V_Q(x)\delta(x-q)}{\partial(\partial^2 S/\partial x_i\partial x_j)} + \cdots. \tag{2.9}$$

The first (delta-function) terms on the right-hand sides are what we expect for the current due to a point particle. We shall not need J_i but it is useful to note, using properties of the delta function together with (2.5) and the result

$$\frac{\delta V_Q}{\delta S(x)} = \frac{\partial\Delta_i}{\partial x_i}, \tag{2.10}$$

that the current components indeed obey the continuity equation:

$$\frac{\partial J_0}{\partial t} + \partial_i J_i = 0. \tag{2.11}$$

The energy–momentum complex of the composite may be computed from the Lagrangian (2.1) in the standard way [8]. The only components we shall need are the energy and momentum densities:

$$T_{00} = \left(\frac{1}{2}m\dot{q}_i^2 + V(q,t) + V_Q(\psi(q))\right)\delta(x-q) - g_1\left(\frac{1}{m}\partial_i(\rho\partial_i S)\right) - g_2\left(\frac{1}{2m}(\partial_i S)^2 + Q + V\right) \tag{2.12}$$

$$T_{0i} = m\dot{q}_i\delta(x-q) - g_1\partial_i\rho - g_2\partial_i S. \tag{2.13}$$

The total energy $\left(\int T_{00}\,\mathrm{d}^3x\right)$ and momentum $\left(\int T_{0i}\,\mathrm{d}^3x\right)$ are conserved under the same conditions on the external potential as in quantum mechanics [7].

3. Constraints resulting in the de Broglie–Bohm law

In our approach, by the phrase ‘quantum system’ we mean the fields ρ, S, g_1 and g_2, and the particle (q). As indicated above, in order to qualify for this appellation certain basic properties of the composite system must coincide with key ones that are relevant to the conventional concept of a quantum system. For a given wavefunction obeying (2.2) and (2.3), we seek constraints on equations (2.4)–(2.6) and their solutions for which the current and energy–momentum complexes of the composite coincide with the usual values of these quantities for the Schrödinger field. It turns out that only three components are sufficient to fix the particle law in terms of ψ.

The Schrödinger field current components are

$$J_0 = \rho, \qquad J_i = \frac{1}{m}\rho\partial_i S. \tag{3.1}$$

Equating J_0 with (2.7), therefore, gives

$$g_2 = -\rho + \delta(x-q). \tag{3.2}$$

The energy[1] and momentum densities of the Schrödinger field are

$$T_{00} = \rho\left(\frac{1}{2m}(\partial_i S)^2 + Q + V\right) \tag{3.3}$$

[1] The expression we use for the energy density involves one of the infinite class [9] of ‘local kinetic energies’, all of whose global values agree. This latitude is a partial expression of the freedom to add a divergence to the energy–momentum complex.

$$T_{0i} = \rho \partial_i S. \tag{3.4}$$

Setting (3.4) equal to (2.13) and using (3.2) gives

$$(m\dot{q}_i - \partial_i S)\delta(x - q) = g_1 \partial_i \rho. \tag{3.5}$$

Suppose $g_1 \neq 0$. Dividing by g_1, integrating over all x and using $\rho \to 0$ as $x_i \to \infty$, (3.5) gives

$$m\dot{q}_i - \partial_i S|_{x=q(t)} = 0. \tag{3.6}$$

It follows that $g_1 \partial_i \rho = 0$ so $g_1 = 0$ when $\partial_i \rho \neq 0$. Either way, (3.6) holds and hence we obtain as a constraint the de Broglie–Bohm equation (1.1). Lastly, we set (3.3) equal to (2.12) and using (3.2) and (3.6) we obtain

$$(V_Q - Q)\delta(x - q) - g_1 \left(\frac{1}{m} \partial_i (\rho \partial_i S) \right) = 0. \tag{3.7}$$

A similar argument to that just given implies that $V_Q = Q$ and $g_1 \dot{\rho} = 0$. Therefore, expressions (2.9) and (2.10) vanish. The conditions we have obtained on g_1 are independent of the particle coordinates and we may set this function to any convenient value that obeys these conditions and the field equations (2.4) and (2.5); a satisfactory choice is $g_1 = 0$.

Given the functions ρ and S, we have derived a set of conditions on g_1, g_2, q and V_Q. These are consistent constraints on equations (2.4)–(2.6). In particular, the particle equation (2.6) with $V_Q = Q$ is implied by (2.3) and (3.6).

4. Comments

In a proper formulation of the dynamics underlying the proposal of de Broglie and Bohm, one treats the particle and wavefunction, and other necessary entities, as partners interacting according to the principles of analytical mechanics (subject to the no back-reaction postulate). Our aim has been to offer an alternative to the statistical argument for the particle law of motion by connecting the latter's justification with the requirement of physical consistency of the model. In fact, a few reasonable conditions are sufficient to guarantee that the law is that of de Broglie and Bohm. Actually, since a gradient flow is preserved in time when a particle dynamics admits an acceleration potential, if we had just established that the quantum influence on the particle is mediated by the quantum potential the constraint (1.1) needs to be imposed only on the initial velocity. The de Broglie–Bohm theory is not, therefore, essentially a 'first-order' dynamical theory [7]. The great significance of the kinematical constraint may be gauged by consideration of the classical limit, defined by negligible quantum potential. For, although from (2.6) we obtain Newton's second law for the particle, the shadow of quantum mechanics persists through the restriction on the initial conditions, a requirement that is alien to the classical description.

As will be discussed elsewhere, the fluid-dynamical version of this theory shows striking divergences from classical fluid models. Further areas to examine, in the context of more general models, are the possibility of deducing information about the particle, such as its mass, and investigation of particle laws different from that of de Broglie and Bohm (yet compatible with $|\psi|^2$). In particular, it will be necessary to include a quantum vector potential in the extension of the theory to spin $1/2$ [2]. With reference to the many attempts that have been made to develop relativistic versions of the trajectory theory, our treatment indicates that the effort to derive future-causal vectors from the pure quantum formalism may be beside the point. In the full canonical approach the particle law is additional to the wave law and hence its properties need not faithfully reproduce those of, e.g., the quantal current. It is known, for example, that the covariant quantum formalism may be derived from a non-covariant trajectory structure [10].

Acknowledgment

The author gratefully acknowledges the support of a Leverhulme Trust Research Fellowship.

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